Rainy day notes on Kaluza-Klein reductions:

Doubtless we cannot see that other higher Spaceland now, because we have no eye in our stomachs.

— Edwin A. Abbott
Flatland: A Romance of Many Dimensions

Follow, I enter my dimension.

— Gojira
Vacuity from "The Way of All Flesh"

A not insignificant fraction of contemporary theoretical physicists (most notably string and M-theorists) operates under the assumption that the dimension of spacetime is bigger than four. Since (most?) people only experience four dimensions in everyday life, the extra dimensions that are present in these theories can not be infinite in extent. Rather, they have to be compact and their size should be small enough so that they escape detection with current experimental/observational means. Kaluza-Klein (KK) theory deals with figuring out which shape and size extra dimensions can have such that higher-dimensional theories can still be consistent with the four-dimensional physics that we all know and love. Extracting effective four-dimensional physics from a higher-dimensional theory is done by a procedure called "Kaluza-Klein reduction".

In these notes, you will see how to perform a KK reduction in the simplest case, in which there is only one extra dimension that is shaped like a circle. The concept of a KK circle reduction of a field theory will first be illustrated in section 1 in a toy model, namely a massless scalar field in Minkowski spacetime. After this, it will be shown how this is extended to General Relativity in section 2. The scalar field toy model will make clear how an extra compact dimension could in theory reveal itself to a lower-dimensional observer. It will in particular show that particles with non-zero momentum in the extra dimension constitute a tower of particles that are massive from the lower-dimensional point of view. Section 3 will offer a different way of seeing this, by examining the worldline action of a particle that moves in a background spacetime with one extra circular dimension.

In the rest of these notes, it will be assumed that spacetime has $D+1$ dimensions, one of which is compact. This compact direction is taken to be spatial and to have the shape of a circle S^1 with radius L. [1] In section 1, the spacetime will be flat, whereas in sections 2 and 3 it can be arbitrarily curved. In any case, the $(D+1)$-dimensional spacetime will be equipped with a set of local coordinates x^M, $M = 0, 1, \cdots, D$. These coordinates will be split in D-dimensional ones x^μ, $\mu = 0, 1, \cdots, D-1$, and the remaining coordinate z that parametrizes the circle S^1:

$$x^M = \{x^\mu, z\}\,. \tag{0.1}$$

Since z is a coordinate on the S^1, it is periodically identified as follows:

$$z \sim z + 2\pi L\,. \tag{0.2}$$

Fields and objects that live in $D+1$ dimensions will be denoted with a hat, e.g., $\hat{g}_{MN}$ stands for a $(D+1)$-dimensional metric, while D-dimensional ones will be unhatted, e.g., $g_{\mu\nu}$ is a D-dimensional metric. [2]

The best way to learn what KK reductions are and how to do them is by actually doing them. For this reason, these notes contain quite a few exercises. They have however been cobbled together rather hastily and have not been thoroughly checked for correctness, so the first exercise is the following:

[1] These notes stick to reductions along spatial directions because they aim to be well-behaved and to minimize the amount of weirdness. One can also consider KK reductions along timelike and null directions. The former lead to Euclidean lower-dimensional theories without time, so they are a bit weird. KK reductions along null directions give, perhaps a bit surprisingly, rise to non-relativistic lower-dimensional theories. Also a bit funky...

[2] Literature involving KK reductions can be quite the festival of hats, tildes, primes and other accents. In these notes, I will do my best to refrain from indulging in notational pedantry and I will try to stick to just hats.

Exercise 1:

- Find all mistakes and let me know...

  Before starting, let me just point out the references used to write up these notes:

  - Chris Pope's lecture notes on "Kaluza-Klein Theory" that can be found on `https://people.tamu.edu/~c-pope/ihplec.pdf`. Sections 1.1 and 1.2 were used for the first two sections of these notes. Pope's lecture notes contain much more material (and deal with e.g., torus and sphere reductions of supergravity theories). They are excellent.
  - For the third section of these notes, I used section 15.2.3 of (the second edition of) Tomas Ortin's "Gravity and Strings" (published by Cambridge University Press). In general this book is a useful, encyclopedic reference on all things that concern supergravity solutions and their relation to string theory. The first sections also contain a wealth of advanced General Relativity material. Really good book, but unfortunately it uses a set of conventions that makes many people cry in their sleep. Also, this reference has zero respect for footnote 2.

1 KK reductions: what and why?

In a nutshell, a KK reduction from $D+1$ to D dimensions along a circle S^1 (with coordinate z), consists of choosing a clever parametrization of the $(D+1)$-dimensional fields in terms of D-dimensional ones that are z-independent. This parametrization is then plugged in whatever $(D+1)$-dimensional action, equations of motion,... one is interested in, to get an effective D-dimensional description of the physics under consideration. The "reduction" [3] part of KK reduction refers to the assumption that all $(D+1)$-dimensional fields are independent of the compact coordinate z. To see why this is physically sensible, let us consider a toy model, namely a massless scalar field $\hat{\phi}$ that lives in $(D+1)$-dimensional Minkowski spacetime and thus obeys the massless Klein-Gordon equation:

$$\partial^M \partial_M \hat{\phi} = 0\,. \tag{1.1}$$

Since the compact coordinate z is periodically identified as in (0.2), one should require the field $\hat{\phi}$ to be periodic in z with period $2\pi L$. One can then Fourier expand $\hat{\phi}$ as:

$$\hat{\phi}(x^\mu, z) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu)\mathrm{e}^{\mathrm{i}nz/L}\,. \tag{1.2}$$

Comparing this Fourier expansion with a standard decomposition of the Minkowski scalar in plane waves $\mathrm{e}^{\mathrm{i}p_M x^M}$, you can see that the momentum p_z in the z-direction is discrete instead of continuous:

$$p_z = \frac{n}{L}\,. \tag{1.3}$$

Plugging the expansion (1.2) in the Klein-Gordon equation (1.1) gives:

$$\partial^M \partial_M \hat{\phi} = \partial^\mu \partial_\mu \hat{\phi} + \partial_z^2 \hat{\phi} = \sum_{n\in\mathbb{Z}} \left(\partial^\mu \partial_\mu \phi_n(x^\nu) - \frac{n^2}{L^2}\phi_n(x^\nu) \right) \mathrm{e}^{\mathrm{i}nz/L} = 0\,. \tag{1.4}$$

The last equality can only hold iff

$$\partial^\mu \partial_\mu \phi_n(x^\nu) - \frac{n^2}{L^2}\phi_n(x^\nu) = 0 \qquad\qquad \forall n\in\mathbb{Z}\,. \tag{1.5}$$

One thus sees that each Fourier mode $\phi_n(x^\mu)$ corresponds to a massive D-dimensional Klein-Gordon scalar with mass M_n given by

$$M_n = \frac{|n|}{L}\,. \tag{1.6}$$

[3] Running a KK reduction in reverse to embed a D-dimensional theory into a higher-dimensional one, is called "oxidation". Who said that theoretical physicists lack a sense of humour?

Note in particular that the zero mode $\phi_0(x^\mu)$ is massless.

In the presence of a compact direction, a $(D+1)$-dimensional massless Klein-Gordon scalar $\hat{\phi}(x^M)$ can thus be interpreted as a D-dimensional massless scalar $\phi_0(x^\mu)$ and a tower of massive ones $\phi_n(x^\mu)$ $(n \neq 0)$. When the size L of the compact direction is very small, the masses (1.6) of the $\phi_n(x^\mu)$ are very large. One can then imagine a situation in which L is small enough that these masses are so large that the particles associated with the $\phi_{n\neq 0}(x^\mu)$ can not be created in current particle physics experiments. These so-called *"Kaluza-Klein excitations"* can then safely be ignored when probing physics at low energies. As long as one is only interested in low-energy physics, one can thus restrict one's attention to only the massless mode $\phi_0(x^\mu)$, effectively setting

$$\hat{\phi} = \phi_0(x^\mu)\,, \tag{1.7}$$

i.e., assuming that $\hat{\phi}$ is independent of z. This process of compactification, along with truncation to the massless sector is what is called KK reduction.

2 Einstein's dream (or nightmare...): a how-to on KK reductions

Above, I mentioned that a KK reduction consists of assuming that all higher-dimensional fields are z-independent, as well as choosing a clever parametrization of the higher-dimensional fields in terms of lower-dimensional ones. Such a z-independent clever parametrization is often called a *"Kaluza-Klein Ansatz"*. For the scalar field example of the previous section, the only such Ansatz one can choose is $\hat{\phi} = \phi(x^\mu)$ and there's not much cleverness involved in that. For the KK reduction of tensor and spinor fields however, picking this Ansatz requires a little more care. Let's have a look at how this is done for a metric field $\hat{g}_{MN}$ and at the same time consider the mother of all KK reductions, namely the KK reduction of the $(D+1)$-dimensional Einstein-Hilbert Lagrangian:

$$\hat{\mathcal{L}}_{\text{EH}} = \sqrt{-\hat{g}}\hat{R}\,. \tag{2.1}$$

As was done for the scalar field, I will assume that the $(D+1)$-dimensional metric $\hat{g}_{MN}$ is independent of the z-coordinate on the S^1. The rationale behind this is the same as for the scalar field: the z-dependence of the higher-dimensional metric is from the lower-dimensional point of view captured by a tower of massive modes that are assumed to be too heavy to be observable by humans in their current evolutionary stage. Splitting the index M into μ and z, one sees that $\hat{g}_{MN}$ has the following components:

$$\hat{g}_{\mu\nu}\,, \qquad\qquad \hat{g}_{\mu z}\,, \qquad\qquad \hat{g}_{zz}\,. \tag{2.2}$$

The first of these looks like a D-dimensional metric, i.e., something that should transform as a symmetric two-tensor under D-dimensional general coordinate transformations (g.c.t.s) of the x^μ coordinates (more on this in a minute!). From the D-dimensional point of view, the index z is just a spectator index that hangs around for the ride and is inert under D-dimensional g.c.t.s. The second $\hat{g}_{\mu z}$ then looks like something that transforms like a one-form under D-dimensional g.c.t.s, while the third $\hat{g}_{zz}$ should behave like a D-dimensional scalar. Naively, one could then think about proposing the following KK Ansatz that expresses the components of the $(D+1)$-dimensional metric $\hat{g}_{MN}$ in D-dimensional ones:

$$\text{naive Ansatz:} \qquad \hat{g}_{\mu\nu} = g_{\mu\nu}(x^\rho)\,, \qquad\qquad \hat{g}_{\mu z} = A_\mu(x^\nu)\ (= \hat{g}_{z\mu})\,, \qquad\qquad \hat{g}_{zz} = \phi(x^\mu)\,. \tag{2.3}$$

While there is philosophically nothing wrong with this parametrization and one could just dump it into (2.1) to get the effective D-dimensional physics described by Einstein-Hilbert gravity in the presence of a small compact dimension, this is not the best of ideas. Philosophers are not known for pursuing clarity in calculations and formulas and the result of using (2.3) would consequently be quite a mess.

To understand why the Ansatz (2.3) leads to a mess and how one can come up with a better one, one should have a look at symmetries. [4] The $(D+1)$-dimensional Einstein-Hilbert action (2.1) is invariant under g.c.t.s, which infinitesimally act on $\hat{g}_{MN}$ as:

$$\delta\hat{g}_{MN} = \mathcal{L}_\xi \hat{g}_{MN} = \hat{\xi}^R\partial_R\hat{g}_{MN} + 2\partial_{(M}\hat{\xi}^R\hat{g}_{N)R}\,. \tag{2.4}$$

[4] One should always look at symmetries! Especially when confronted with longish calculations! See exercise 5...

Here, $\hat{\xi}^M$ are the parameters of an infinitesimal g.c.t. (i.e., under such a g.c.t. the coordinates x^M change as $\delta x^M = -\hat{\xi}^M$). They are local, so they depend arbitrarily on the x^M coordinates. Let's then think about the symmetries that the lower-dimensional theory should have. The D-dimensional theory that arises upon performing the KK reduction should inherit part of the invariance of its $(D+1)$-dimensional parent under the $(D+1)$-dimensional g.c.t.s (2.4), but it can not inherit invariance under the full $(D+1)$-dimensional g.c.t.s. Indeed, when performing the KK reduction, all we do is plug in whatever KK Ansatz we pick in a $(D+1)$-dimensional action, equations of motion,... The resulting D-dimensional theory will then only be invariant under that part of the $(D+1)$-dimensional symmetries that preserves our chosen KK Ansatz! In our case, our KK Ansatz assumes that $\hat{g}_{MN}$ is z-independent. We should thus also restrict ourselves to symmetry parameters $\hat{\xi}^M$ that do not induce a dependence on z in the metric. In other words, we should require the $\hat{\xi}^M$ to be such that $\delta\hat{g}_{MN}$ is z-independent:

$$\partial_z\left(\delta\hat{g}_{MN}\right) = \partial_z\hat{\xi}^R\partial_R\hat{g}_{MN} + 2\partial_z\partial_{(M}\hat{\xi}^R\hat{g}_{N)R} = 0\,. \tag{2.5}$$

Here, I used that $\partial_z\hat{g}_{MN} = 0$. You can convince yourself that this requirement restricts the $\hat{\xi}^M$ parameters to be of the form:

$$\hat{\xi}^\mu = \xi^\mu(x^\nu)\,, \qquad \hat{\xi}^z = cz + \lambda(x^\mu)\,, \qquad \text{with } c \text{ a constant}\,. \tag{2.6}$$

For $\hat{\xi}^z$ this convincing is done in Appendix A.

Exercise 2:

> Try to come up with whatever dirty argument that ensures that you don't lose sleep over the fact that $\hat{\xi}^\mu = \xi^\mu(x^\nu)$.

It should not come as a big surprise that the parameters ξ^μ will be those of infinitesimal D-dimensional g.c.t.s. What kind of D-dimensional symmetry transformation the parameter λ (associated to x^μ-dependent shifts of the z-coordinate) corresponds to, will become clear in a minute. (Can you perhaps already guess?)

Let us now have a look at how $g_{\mu\nu}$ in our naive Ansatz (2.3) transforms under the D-dimensional transformations parametrized by $\xi^\mu(x^\nu)$ and $\lambda(x^\mu)$.[5] Ideally, we should find that $g_{\mu\nu}$ transforms like a symmetric two-tensor under the D-dimensional infinitesimal g.c.t.s with parameters ξ^μ. Let's see whether this is what we get by just writing out (2.4) for the index choice $M=\mu$, $N=\nu$ and using (2.6), as well as the naive Ansatz (2.3)

$$\begin{aligned}\delta\hat{g}_{\mu\nu} &= \hat{\xi}^\rho\partial_\rho\hat{g}_{\mu\nu} + 2\partial_{(\mu}\hat{\xi}^\rho\hat{g}_{\nu)\rho} + 2\partial_{(\mu}\hat{\xi}^z\hat{g}_{\nu)z}\\ \Rightarrow \qquad \delta g_{\mu\nu} &= \xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho} + 2A_{(\mu}\partial_{\nu)}\lambda\,.\end{aligned} \tag{2.7}$$

The first two terms of $\delta g_{\mu\nu}$ indeed reproduce the usual transformation rule of the symmetric two-tensor $g_{\mu\nu}$ under D-dimensional infinitesimal g.c.t.s. The last term that involves the parameter λ does however not correspond to a transformation that we expect for a symmetric two-tensor. This term is the reason why the KK reduction with the naive Ansatz (2.3) leads to a jumble of $g_{\mu\nu}$s, A_μs and ϕs: the lower-dimensional fields in this Ansatz do not have conventional lower-dimensional transformation rules and we can thus not expect the lower-dimensional theory for these fields to be in a form that is recognizable as something well-known.

With this knowledge, let us now have a look at the following better Ansatz:

$$\hat{g}_{\mu\nu} = \mathrm{e}^{2\alpha\varphi}g_{\mu\nu} + \mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\mathcal{A}_\nu\,, \qquad \hat{g}_{\mu z} = \mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\,, \qquad \hat{g}_{zz} = \mathrm{e}^{2\beta\varphi}\,, \tag{2.8}$$

where it is understood that the D-dimensional fields $g_{\mu\nu}$, $\mathcal{A}_\mu$ and φ are z-independent and where α, β are constants (to be fixed later). Note that we should take $\beta \neq 0$, since we would like to parametrize $\hat{g}_{MN}$ in the most general way possible (and $\hat{g}_{zz} = 1$ is definitely not very general). In matrix notation, this looks like

$$\hat{g}_{MN} = \begin{matrix} \\ \mu \\ z\end{matrix}\begin{pmatrix} \overset{\nu}{\mathrm{e}^{2\alpha\varphi}g_{\mu\nu} + \mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\mathcal{A}_\nu} & \overset{z}{\mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu} \\ \mathrm{e}^{2\beta\varphi}\mathcal{A}_\nu & \mathrm{e}^{2\beta\varphi}\end{pmatrix}, \tag{2.9}$$

[5] In what follows, I'll ignore the transformation with the parameter c, so I will assume that $\partial_z\hat{\xi}^z = 0$. It leads to a global dilatation invariance of the D-dimensional theory. Usually, it is the local symmetries that are most handy when it comes to repackaging a mess into something that can be wrapped in gift paper, so I'll focus on those.

and in terms of the $(D+1)$-dimensional line element we have

$$\mathrm{d}\hat{s}^2 = \mathrm{e}^{2\alpha\varphi}\mathrm{d}s^2 + \mathrm{e}^{2\beta\varphi}\left(\mathrm{d}z + \mathcal{A}_\mu \mathrm{d}x^\mu\right)^2 . \tag{2.10}$$

We can now check that $g_{\mu\nu}$, $\mathcal{A}_\mu$ and φ have decent D-dimensional transformation rules under the symmetries with parameters ξ^μ and λ. First of all, specifying $M = N = z$ in (2.4), and using (2.6) as well as the Ansatz (2.8), leads to

$$\begin{aligned} &\delta\hat{g}_{zz} = \hat{\xi}^\mu\partial_\mu\hat{g}_{zz} \qquad \Rightarrow \qquad \delta\mathrm{e}^{2\beta\varphi} = \xi^\mu\partial_\mu\mathrm{e}^{2\beta\varphi} \qquad \Rightarrow \qquad 2\beta\mathrm{e}^{2\beta\varphi}\delta\varphi = 2\beta\mathrm{e}^{2\beta\varphi}\xi^\mu\partial_\mu\varphi \\ \Rightarrow \quad &\delta\varphi = \xi^\mu\partial_\mu\varphi . \end{aligned} \tag{2.11}$$

The field φ indeed transforms as a scalar under the D-dimensional g.c.t.s with parameter ξ^μ. Specifying $M = \mu$, $N = z$ in (2.4), and using (2.6) as well as the Ansatz (2.8), gives

$$\begin{aligned} \delta\hat{g}_{\mu z} &= \hat{\xi}^\nu\partial_\nu\hat{g}_{\mu z} + \hat{g}_{z\nu}\partial_\mu\hat{\xi}^\nu + \hat{g}_{zz}\partial_\mu\hat{\xi}^z \\ \Rightarrow \quad \delta\left(\mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\right) &= \xi^\nu\partial_\nu\left(\mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\right) + \mathrm{e}^{2\beta\varphi}\mathcal{A}_\nu\partial_\mu\xi^\nu + \mathrm{e}^{2\beta\varphi}\partial_\mu\lambda \\ \Rightarrow \quad 2\beta\mathrm{e}^{2\beta\varphi}\mathcal{A}_\mu\delta\varphi + \mathrm{e}^{2\beta\varphi}\delta\mathcal{A}_\mu &= 2\beta\mathrm{e}^{2\beta\varphi}\xi^\nu\partial_\nu\varphi\mathcal{A}_\mu + \mathrm{e}^{2\beta\varphi}\left(\xi^\nu\partial_\nu\mathcal{A}_\mu + \mathcal{A}_\nu\partial_\mu\xi^\nu\right) + \mathrm{e}^{2\beta\varphi}\partial_\mu\lambda \\ \Rightarrow \quad \delta\mathcal{A}_\mu &= \xi^\nu\partial_\nu\mathcal{A}_\mu + \mathcal{A}_\nu\partial_\mu\xi^\nu + \partial_\mu\lambda . \end{aligned} \tag{2.12}$$

In the last step, we used the transformation rule (2.11) of φ. The first two terms of $\delta\mathcal{A}_\mu$ tell us that $\mathcal{A}_\mu$ transforms as a one-form under D-dimensional g.c.t.s with parameters ξ^μ. From the last term, we see that $\mathcal{A}_\mu$ also transforms under the symmetry with parameter λ. Unlike what happened for the naive $g_{\mu\nu}$ in (2.7), this λ-transformation is something that we know very well and love a lot: λ corresponds to the parameter of a D-dimensional abelian gauge symmetry, [6] under which $\mathcal{A}_\mu$ transforms as a gauge field.

Exercise 3:

Take $M = \mu$ and $N = \nu$ in (2.4) and check that $g_{\mu\nu}$ transforms like a symmetric two-tensor under the D-dimensional g.c.t.s:

$$\delta g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho} . \tag{2.13}$$

Summarizing, we find the following transformation rules for the D-dimensional fields:

$$\delta g_{\mu\nu} = \xi^\rho\partial_\rho g_{\mu\nu} + 2\partial_{(\mu}\xi^\rho g_{\nu)\rho} , \qquad \delta\mathcal{A}_\mu = \xi^\nu\partial_\nu\mathcal{A}_\mu + \mathcal{A}_\nu\partial_\mu\xi^\nu + \partial_\mu\lambda , \qquad \delta\varphi = \xi^\mu\partial_\mu\varphi . \tag{2.14}$$

Before continuing, let us briefly comment on the physical/geometric meaning of $g_{\mu\nu}$ and φ. The former corresponds to a D-dimensional metric field. The meaning of φ becomes clear by noting that the length of the internal circle S^1 at a fixed D-dimensional spacetime point with coordinates x^μ is given by

$$\int_0^{2\pi L} \mathrm{d}z\, \mathrm{e}^{\beta\varphi(x^\mu)} = 2\pi L\mathrm{e}^{\beta\varphi(x^\mu)} . \tag{2.15}$$

We thus see that the size of the internal circle is determined by φ. In a KK reduction, we allow the size of this S^1 to (smoothly) change, as we move through the lower-dimensional space-time. This is why the size of the S^1 is not captured by a constant but rather by a D-dimensional scalar field φ. The field φ is usually called the *"dilaton"*. [7] The one-form $\mathcal{A}_\mu$ is universally known as the *"Kaluza-Klein vector"*. The physical

[6] This is in fact a U(1) gauge symmetry (as opposed to a non-compact $\mathbb{R}$ gauge symmetry). The reason is that the parameter λ should be periodically identified, since it corresponds to x^μ-dependent shifts of the periodic coordinate z.

[7] Strictly speaking, one should call this the "KK dilaton" to not confuse it with the "string dilaton", another scalar field that appears in all string theories. Before you start ranting about how physicists always abuse the same terminology in slightly different contexts, let me note that there is one instance where there is a "dilaton equivalence principle", i.e., where KK dilaton = string dilaton. The string dilaton of type IIA string theory and its low-energy effective supergravity theory does correspond to the KK dilaton that arises upon KK reduction of 11-dimensional supergravity, the low-energy approximation of M-theory.

meaning of the KK vector $\mathcal{A}_\mu$ and its associated abelian gauge symmetry is not obvious in the KK reductions that we consider here. This meaning would become more clear when we would not do the reduction part of KK reduction and instead keep all massive modes in the analogue of the Fourier expansion (1.2) for $\hat{g}_{MN}$. In that case, one would find that all these massive modes are charged under the abelian gauge symmetry with parameter λ, i.e., that they transform with phase factors $\mathrm{e}^{in\lambda/L}$. You can see this in the scalar toy model, by looking at a term $\phi_n(x^\mu)\mathrm{e}^{inz/L}$ in the Fourier series (1.2) that contains a massive mode. The effect of the shift $z \to z + \lambda(x^\mu)$ in $\mathrm{e}^{inz/L}$ can be re-interpreted as a transformation $\phi_n(x^\mu) \to \mathrm{e}^{in\lambda/L}\phi_n(x^\mu)$. Since the massive modes are subjected to such a U(1) gauge transformation, their derivatives have to be covariantized with respect to a gauge field for this symmetry, but this is just the KK vector $\mathcal{A}_\mu$. We will not go into further detail about this here. We will however discuss the fact that particles with non-zero momentum n/L along the z-direction (that correspond to modes with mass $|n|/L$) couple non-trivially to $\mathcal{A}_\mu$ from a different point of view later in these notes.

In principle, one can now do the KK reduction of the Einstein-Hilbert action (2.1) by plugging in the Ansatz (2.8) in (2.1) and massaging the result until one ends up with a D-dimensional Lagrangian that is in a recognizable form. Like everything in life, it turns out that this computation is easier in the Vielbein formulation instead of in the metric formulation. So, instead of using a metric $\hat{g}_{MN}$, we will work with a Vielbein $\hat{e}_M{}^A$ that is related to $\hat{g}_{MN}$ by

$$\hat{g}_{MN} = \hat{e}_M{}^A \eta_{AB} \hat{e}_N{}^B \,. \tag{2.16}$$

Here, the so-called "flat" index $A = 0, 1, \cdots, D$ and η_{AB} is the $(D+1)$-dimensional Minkowski metric in the mostly-plus convention (the only sane Minkowski metric convention). These flat indices will be liberally raised and lowered with η_{AB} and its inverse η^{AB}. We will use $\hat{e}_A{}^M$ to denote the inverse of $\hat{e}_M{}^A$:

$$\hat{e}_M{}^A \hat{e}_A{}^N = \delta_M^N \,, \qquad\qquad \hat{e}_A{}^M \hat{e}_M{}^B = \delta_A^B \,. \tag{2.17}$$

The inverse Vielbein is often used as a notational device to "turn curved M, N indices into flat A, B ones". As an example, consider a one-form $\hat{A}_M$ and a two-form $\hat{B}_{MN}$. Often, one would see these objects with flat A, B indices, i.e., as $\hat{A}_A$, $\hat{B}_{AB}$, $\hat{B}_{MA}$, $\hat{B}_{AN}$. These are not new objects; rather they are defined to be related to $\hat{A}_M$ and $\hat{B}_{MN}$ as follows:

$$\hat{A}_A \equiv \hat{e}_A{}^M \hat{A}_M \,, \qquad \hat{B}_{AB} \equiv \hat{e}_A{}^M \hat{e}_B{}^N \hat{B}_{MN} \,, \qquad \hat{B}_{AN} \equiv \hat{e}_A{}^M \hat{B}_{MN} \,, \qquad \hat{B}_{MA} \equiv \hat{e}_A{}^N \hat{B}_{MN} \,. \tag{2.18}$$

Instead of the Levi-Civita connection $\hat{\Gamma}^M_{NR}$, we will use the Levi-Civita spin-connection $\hat{\omega}_M{}^{AB}$ that is given in terms of the (inverse) Vielbein and its first-order derivatives by the following expression:

$$\hat{\omega}_M{}^{AB} = 2\hat{e}^{[A|N|}\partial_{[M}\hat{e}_{N]}{}^{B]} - \hat{e}^{[A|N|}\hat{e}^{B]R}\hat{e}_{MC}\partial_N \hat{e}_R{}^C \,. \tag{2.19}$$

The Ricci scalar is in the Vielbein formulation given by

$$\hat{R} = \eta^{AB}\hat{R}_{AB} \,, \tag{2.20}$$

with

$$\hat{R}_{AB} = -\hat{e}_A{}^M \hat{e}_C{}^N \left(2\,\partial_{[M}\hat{\omega}_{N]}{}^C{}_B + 2\,\hat{\omega}_{[M}{}^{CD}\hat{\omega}_{N]DB}\right) \,. \tag{2.21}$$

To perform the KK reduction in the Vielbein formulation, we first split the flat index A into

$$A = \{a, \underline{z}\} \,, \qquad\qquad \text{with } \; a = 0, 1, \cdots, D-1 \,, \tag{2.22}$$

in analogy to how the index M was split in μ and z. We then choose a KK Ansatz for the Vielbein $\hat{e}_M{}^A$ that reproduces the Ansatz (2.8) for the metric $\hat{g}_{MN}$, i.e., that is such that (2.16) is fulfilled with $\hat{g}_{MN}$ given by (2.8). A convenient choice [8] of KK Ansatz for the Vielbein is given by:

$$\hat{e}_M{}^A = \begin{matrix} \\ \mu \\ z \end{matrix} \begin{pmatrix} \overset{a}{\mathrm{e}^{\alpha\varphi}e_\mu{}^a} & \overset{\underline{z}}{\mathrm{e}^{\beta\varphi}\mathcal{A}_\mu} \\ 0 & \mathrm{e}^{\beta\varphi} \end{pmatrix} , \tag{2.23}$$

[8] Choosing a Vielbein for a particular metric can be done in an infinite number of ways. In particular, two Vielbeine $\hat{e}_M{}^A$ and $\hat{e}'_M{}^A$ that are related via $\hat{e}'_M{}^A = \Lambda^A{}_B\hat{e}_M{}^B$, where $\Lambda^A{}_B$ is the matrix of a local (so arbitrarily x^M-dependent) Lorentz transformation, give rise to the same metric $\hat{g}_{MN}$. The upper-triangular choice picked here, for which $\hat{e}_z{}^a = 0$, can be viewed as a gauge-fixing for the Lorentz transformations with parameters $\Lambda^a{}_{\underline{z}}$. Indeed, requiring that $\hat{e}_z{}^a = 0$ is preserved under local Lorentz transformations, gives $0 = \Lambda^a{}_b\hat{e}_z{}^b + \Lambda^a{}_{\underline{z}}\hat{e}_z{}^{\underline{z}} = \mathrm{e}^{\beta\varphi}\Lambda^a{}_{\underline{z}}$, from which $\Lambda^a{}_{\underline{z}} = 0$ follows.

where all quantities in the matrix on the right-hand-side are independent of z. Here, $e_\mu{}^a$ is a Vielbein for the D-dimensional metric $g_{\mu\nu}$ that appears in (2.8):

$$e_\mu{}^a e_\nu{}^b \eta_{ab} = g_{\mu\nu}\,. \tag{2.24}$$

One then indeed has that (2.16) holds. The corresponding KK Ansatz for the inverse Vielbein $\hat{e}_A{}^M$ is given by:

$$\hat{e}_A{}^M = \begin{matrix} \\ a \\ \underline{z} \end{matrix} \begin{pmatrix} \mu & z \\ \mathrm{e}^{-\alpha\varphi} e_a{}^\mu & -\mathrm{e}^{-\alpha\varphi} e_a{}^\mu \mathcal{A}_\mu \\ 0 & \mathrm{e}^{-\beta\varphi} \end{pmatrix}, \tag{2.25}$$

where $e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu$ and $e_a{}^\mu e_\mu{}^b = \delta_a^b$, so that indeed $\hat{e}_M{}^A \hat{e}_A{}^N = \delta_M^N$ and $\hat{e}_A{}^M \hat{e}_M{}^B = \delta_A^B$. Notice how much easier and more natural (2.23) looks when compared to (2.8)! The Vielbein formulation truly aids in living a happier and more fulfilled life.

You can now perform the KK reduction of (2.1) yourself by going through the following set of exercises! First, reduce the spin-connection $\hat{\omega}_M{}^{AB}$:

Exercise 4:

Plug the KK Ansätze (2.23) and (2.25) in the expression (2.19) for the spin-connection $\hat{\omega}_M{}^{AB}$ to obtain:

$$\begin{aligned}
\hat{\omega}_\mu{}^{ab} &= \omega_\mu{}^{ab} - 2\alpha\partial^{[a}\varphi e_\mu{}^{b]} - \frac{1}{2}\mathrm{e}^{2(\beta-\alpha)\varphi}\mathcal{A}_\mu \mathcal{F}^{ab}\,,\\
\hat{\omega}_z{}^{ab} &= -\frac{1}{2}\mathrm{e}^{2(\beta-\alpha)\varphi}\mathcal{F}^{ab}\,,\\
\hat{\omega}_\mu{}^{a\underline{z}} &= -\frac{1}{2}\mathrm{e}^{(\beta-\alpha)\varphi}\mathcal{F}^a{}_\mu - \beta\mathrm{e}^{(\beta-\alpha)\varphi}\mathcal{A}_\mu\partial^a\varphi\,,\\
\hat{\omega}_z{}^{a\underline{z}} &= -\beta\mathrm{e}^{(\beta-\alpha)\varphi}\partial^a\varphi\,.
\end{aligned} \tag{2.26}$$

Here,

$$\omega_\mu{}^{ab} = 2e^{[a|\nu|}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{[a|\nu|}e^{b]\rho}e_{\mu c}\partial_\nu e_\rho{}^c\,, \tag{2.27}$$

is the D-dimensional spin-connection (constructed from the Vielbein $e_\mu{}^a$ and its inverse $e_a{}^\mu$) and

$$\mathcal{F}_{\mu\nu} = 2\,\partial_{[\mu}\mathcal{A}_{\nu]}\,, \tag{2.28}$$

is the field strength of the KK vector $\mathcal{A}_\mu$. In (2.26) (as well as in everything that follows), curved μ, ν indices have been turned into flat a, b ones by contraction with $e_a{}^\mu$, e.g.:

$$\partial^a\varphi \equiv e^{a\mu}\partial_\mu\varphi\,, \qquad \mathcal{F}^{ab} \equiv e^{a\mu}e^{b\nu}\mathcal{F}_{\mu\nu}\,, \qquad \mathcal{F}^a{}_\mu \equiv e^{a\nu}\mathcal{F}_{\nu\mu}\,. \tag{2.29}$$

Next, we need the KK reduction of the Ricci scalar $\hat{R}$. Splitting the flat indices in (2.20)

$$\hat{R} = \eta^{ab}\hat{R}_{ab} + \eta^{\underline{zz}}\hat{R}_{\underline{zz}} = \eta^{ab}\hat{R}_{ab} + \hat{R}_{\underline{zz}}\,, \tag{2.30}$$

we see that we need the reductions of $\hat{R}_{ab}$ and $\hat{R}_{\underline{zz}}$ for this. This is the content of the following:

Exercise 5:

Use (2.26) and the KK Ansätze (2.23), (2.25) for the (inverse) Vielbein in (2.21), to get

$$\begin{aligned}
\hat{R}_{ab} &= \mathrm{e}^{-2\alpha\varphi}R_{ab} - (\beta + (D-2)\alpha)\,\mathrm{e}^{-2\alpha\varphi}\nabla_a\partial_b\varphi - \alpha\mathrm{e}^{-2\alpha\varphi}\eta_{ab}\nabla_c\partial^c\varphi - \frac{1}{2}\mathrm{e}^{2(\beta-2\alpha)\varphi}\mathcal{F}_a{}^c\mathcal{F}_{bc}\\
&\quad + \left((D-2)\alpha^2 + 2\alpha\beta - \beta^2\right)\mathrm{e}^{-2\alpha\varphi}\partial_a\varphi\partial_b\varphi - \alpha\left[(D-2)\alpha + \beta\right]\mathrm{e}^{-2\alpha\varphi}\partial^c\varphi\partial_c\varphi\eta_{ab}\,,\\
\hat{R}_{\underline{zz}} &= -\beta\mathrm{e}^{-2\alpha\varphi}\nabla_a\partial^a\varphi - \beta\left[\beta + (D-2)\alpha\right]\mathrm{e}^{-2\alpha\varphi}\partial^a\varphi\partial_a\varphi + \frac{1}{4}\mathrm{e}^{2(\beta-2\alpha)\varphi}\mathcal{F}^{ab}\mathcal{F}_{ab}\,.
\end{aligned} \tag{2.31}$$

Here,

$$R_{ab} = -e_a{}^{\mu} e_c{}^{\nu} \left(2\,\partial_{[\mu}\omega_{\nu]}{}^c{}_b + 2\,\omega_{[\mu}{}^{cd}\omega_{\nu]db} \right) , \tag{2.32}$$

and the derivative $\nabla_a = e_a{}^{\mu}\nabla_\mu$ is covariantized with respect to the D-dimensional spin-connection $\omega_\mu{}^{ab}$:

$$\nabla_a \partial_b \varphi = e_a{}^{\mu} \left(\partial_\mu \partial_b \varphi + \omega_{\mu b}{}^c \partial_c \varphi \right) . \tag{2.33}$$

If you have some time to spare, you should try this exercise in two ways:

1. Switch off your brain, plug everything in, work out *all* terms and try to write the result in the form (2.31). After finishing this, spend some time in Nature, listening to soothing birdsong, until every bit of existential pain and frustration has left your system and...
2. Start thinking about symmetries and how they can be of therapeutic help for you. We know that the answer has to be invariant under D-dimensional g.c.t.s with parameters ξ^μ and the abelian gauge transformation with parameter λ. In the Vielbein formulation, one will also have invariance under D-dimensional local Lorentz transformations that infinitesimally act on $e_\mu{}^a$ and $e_a{}^\mu$ as

$$\delta e_\mu{}^a = -\lambda^a{}_b e_\mu{}^b , \qquad \delta e_a{}^\mu = -\lambda_a{}^b e_b{}^\mu , \qquad \text{with} \;\; \lambda^{ab} = -\lambda^{ba} . \tag{2.34}$$

   The spin-connection $\omega_\mu{}^{ab}$ transforms like a gauge field for these local Lorentz transformations, i.e., one has that the expression (2.27) transforms under (2.34) as

$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - 2\,\lambda^{[a}{}_c \omega_\mu{}^{|c|b]} . \tag{2.35}$$

   Since we spent quite some time arguing that our KK Ansatz is written in terms of D-dimensional fields that have nice transformation properties under all these symmetries, we expect that our results can be written in a manifestly symmetric way, i.e., in terms of curvatures, covariant derivatives,... that transform nicely under all symmetries. In particular, we expect that any term that contains a "naked" $\omega_\mu{}^{ab}$, i.e., an $\omega_\mu{}^{ab}$ that does not appear under a derivative, will be part of a covariantization of a derivative (e.g., will complement $\partial_a\partial_b\varphi$ to $\nabla_a\partial_b\varphi$) or will turn $2\,\partial_{[\mu}\omega_{\nu]}{}^{ab}$ into a proper curvature. Likewise, a naked $\mathcal{A}_\mu$ can only appear as part of a derivative that is covariant with respect to the abelian gauge transformation with parameter λ. Since only the massive KK modes are charged non-trivially under this gauge transformation and we have truncated them, one can immediately conclude that all terms with a naked $\mathcal{A}_\mu$ have to cancel and that $\mathcal{A}_\mu$ can only occur in a field strength $\mathcal{F}_{\mu\nu}$!

   The upshot of this is that there is no need to compute terms that involve a naked $\omega_\mu{}^{ab}$ or $\mathcal{A}_\mu$! Symmetries guarantee that they end up in covariantizations anyhow! You can thus just ignore these terms and instead compute all remaining terms and covariantize (turn ∂_a into ∇_a and $2\,\partial_{[\mu}\omega_{\nu]}{}^{ab}$ into a proper curvature) what you get as a result at the end. Use the symmetry force in this way to derive the expressions (2.31) and enjoy a strange blend of feelings of frustration and happiness...

We are almost there. We just need to

Exercise 6:

plug (2.31) in (2.30) to get

$$\begin{aligned}\hat{R} = {} & \mathrm{e}^{-2\alpha\varphi} R - \frac{1}{4}\mathrm{e}^{2(\beta-2\alpha)\varphi}\mathcal{F}^{ab}\mathcal{F}_{ab} - 2\left[\alpha+\beta+(D-2)\alpha\right]\mathrm{e}^{-2\alpha\varphi}\nabla_a\partial^a\varphi \\ & - \left[\alpha^2(D-1)(D-2) + 2\beta\left(\beta+(D-2)\alpha\right)\right]\mathrm{e}^{-2\alpha\varphi}\partial^a\varphi\partial_a\varphi .\end{aligned} \tag{2.36}$$

Here, $R = \eta^{ab}R_{ab}$ is the D-dimensional Ricci scalar.

We still need to multiply this with $\sqrt{-\hat{g}}$, which reduces to

$$\sqrt{-\hat{g}} = \det\left(\hat{e}_M{}^A\right) = \mathrm{e}^{(\beta+D\alpha)\varphi}\det\left(e_\mu{}^a\right) = \mathrm{e}^{(\beta+D\alpha)\varphi}\sqrt{-g} . \tag{2.37}$$

We can then finally assemble everything to obtain that the Einstein-Hilbert Lagrangian (2.1) KK reduces to:

$$\sqrt{-\hat{g}}\hat{R} = \sqrt{-g}\Big[\mathrm{e}^{(\beta+(D-2)\alpha)\varphi}R - \frac{1}{4}\mathrm{e}^{(3\beta+(D-4)\alpha)\varphi}\mathcal{F}^{ab}\mathcal{F}_{ab} - 2\left(\alpha+\beta+(D-2)\alpha\right)\mathrm{e}^{(\beta+(D-2)\alpha)\varphi}\nabla_a\partial^a\varphi \\ - \left[\alpha^2(D-1)(D-2) + 2\beta(\beta+\alpha(D-2))\right]\mathrm{e}^{(\beta+(D-2)\alpha)\varphi}\partial^a\varphi\partial_a\varphi\Big] \,. \tag{2.38}$$

We can make this aesthetically more pleasing, by choosing convenient values for the parameters α and β. The first term would correspond to the D-dimensional Einstein-Hilbert action, were it not multiplied with an extra exponential of the dilaton. By choosing

$$\beta = -(D-2)\alpha \,, \tag{2.39}$$

we can make this dilaton exponential disappear, so that the first term reduces to the standard Einstein-Hilbert Lagrangian. With this choice, (2.38) simplifies to

$$\sqrt{-\hat{g}}\hat{R} = \sqrt{-g}\Big[R - \frac{1}{4}\mathrm{e}^{-2(D-1)\alpha\varphi}\mathcal{F}^{ab}\mathcal{F}_{ab} - 2\alpha\nabla_a\partial^a\varphi - \alpha^2(D-1)(D-2)\partial^a\varphi\partial_a\varphi\Big] \,. \tag{2.40}$$

The third term is now a total derivative and can be safely dropped from the Lagrangian. The last term is a kinetic term for a massless scalar field. We can then choose

$$\alpha^2 = \frac{1}{2(D-1)(D-2)} \,, \tag{2.41}$$

so that this kinetic term has the conventional prefactor of 1/2.

Summarizing the above calculation, we find that $(D+1)$-dimensional Einstein-Hilbert gravity on spacetimes with one compact circular direction is at low energies effectively described by the following D-dimensional Lagrangian

$$\mathcal{L}_{\text{EH}-\text{KK}} = \sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\mathrm{e}^{-2(D-1)\alpha\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right] \,, \qquad \text{with} \;\; \alpha = \frac{1}{\sqrt{2(D-1)(D-2)}} \,. \tag{2.42}$$

This is a sum of D-dimensional Einstein-Hilbert gravity, a massless dilaton scalar field coupled to gravity and Maxwell electromagnetism coupled to gravity and the dilaton. When Einstein got wind of this result, he was quickly enamored with it. After he came up with General Relativity, he dreamt obsessively about unifying gravity with electromagnetism and it looked like this was the fulfilment of that dream. All one has to argue for is that one can consistently truncate that dilaton φ and there it is: four-dimensional gravity and electromagnetism are charmingly unified as five-dimensional gravity on a spacetime with one compact direction.

If that sounds too good to be true, it is because it is. It did not take long for people to realize that one can not truncate the dilaton consistently and for Einstein's dream to turn into a lifelong nightmare of failed attempts at unification. To see this, let us have a look at the equations of motion of the Lagrangian (2.42)

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi\right) + \frac{1}{2}\mathrm{e}^{-2(D-1)\alpha\varphi}\left(\mathcal{F}_{\mu\rho}\mathcal{F}_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}\mathcal{F}^{\rho\sigma}\mathcal{F}_{\rho\sigma}\right) \,, \\ \nabla^\mu\left(\mathrm{e}^{-2(D-1)\alpha\varphi}\mathcal{F}_{\mu\nu}\right) &= 0 \,, \\ \nabla_\mu\partial^\mu\varphi &= -\frac{1}{2}(D-1)\alpha\mathrm{e}^{-2(D-1)\alpha\varphi}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \,. \end{aligned} \tag{2.43}$$

The last equation of motion, the one for the dilaton φ, is where the problem is: it has a source term that involves $\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}$ on its right-hand-side and this source term does not vanish when $\varphi = 0$. Setting $\varphi = 0$ thus implies that we also have to require that $\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}$ vanishes and then we definitely can no longer claim that we are dealing with electrodynamics coupled to gravity! If we want to have electrodynamics, we are forced to also keep the dilaton φ. One might object that perhaps Einstein's dream can be saved if the effect of this dilaton on four-dimensional physics is negligible. As a massless scalar, it however gives rise to an extra attractive, long-range, "fifth" force that is most definitely not observed in Nature.

Note that had we made the mistake of taking $\beta = 0$ in the Ansatz (2.8) (see the remark made under (2.8)), we would not have noticed this issue immediately. We would have done the KK reduction without φ and we would have ended up with Einstein-Maxwell gravity, i.e., Einstein-Hilbert gravity coupled to Maxwell electromagnetism. This would however be rather unattractive and even inconsistent from a higher-dimensional perspective. To see why, note that the equations of motion (2.43) (or a set of equivalent equations of motion) can also be obtained by KK reducing the $(D+1)$-dimensional Einstein equations $\hat{R}_{MN} = 0$. The equations of motion of Einstein-Maxwell gravity correspond to the two equations that are obtained by setting $\varphi = 0$ in only the first two equations of (2.43), that correspond to $\hat{R}_{\mu\nu} = 0$ and $\hat{R}_{\mu z} = 0$. From a higher-dimensional point of view, one however also has an extra equation of motion, namely $\hat{R}_{zz} = 0$ and this corresponds to the equation obtained by setting $\varphi = 0$ in the last of (2.43), i.e., this gives the constraint that $\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = 0$. Most of the solutions of the Einstein-Maxwell equations ($\hat{R}_{\mu\nu} = 0$ and $\hat{R}_{\mu z} = 0$) will therefore not be solutions of the higher-dimensional theory, since they do not satisfy the constraint that comes from $\hat{R}_{zz} = 0$! Philosophers would quite rightly grumble about this: the higher-dimensional theory is the more fundamental one and we should therefore be able to properly embed every lower-dimensional phenomenon/solution in the higher-dimensional theory.

At this point, you might start to feel a bit uneasy. If it is so problematic to truncate the dilaton, how do we know that we can consistently truncate the massive modes!? This is another issue where symmetries come to the rescue. Consider a KK theory with massless modes that we collectively denote by χ_0 and a bunch of massless modes, collectively called χ_n (with $n \neq 0$, $n \in \mathbb{Z}$). Let's have look at the equation of motion of a massive mode χ_n. Schematically, this looks like:

$$\mathcal{O}(\chi_n) = \text{source terms}\,. \tag{2.44}$$

Here, $\mathcal{O}(\chi_n)$ is a complicated non-linear differential operator that acts on the mode χ_n and that can in general also depend on the other massive modes and the massless modes χ_0. It does however vanish when the massive modes are set to zero. The source terms on the right-hand-side can likewise depend on all massive and massless modes. Since the theory is symmetric under the abelian gauge symmetry with parameter λ, we know that the expression $\mathcal{O}(\chi_n)$ transforms nicely with a non-trivial phase factor $\mathrm{e}^{ia\lambda}$ under this symmetry (this is how χ_n transforms and any properly covariantized differential operator transforms in the same way). Symmetry then dictates that the source terms on the right-hand-side also transform with the same phase factor. This immediately implies that every source term has to contain at least one massive mode. Indeed, massless modes are invariant under this symmetry and a source term that only consists of massless modes would not transform with the necessary phase factor. We conclude that all source terms vanish when the massive modes are truncated (unlike what happened with the source terms of the dilaton equation of motion when we truncated the dilaton). So, we don't have to worry about the consistency of the truncation of the massive modes, thanks to a slick symmetry argument!

The reduction on a circle S^1 is only the simplest example of a KK reduction. In the literature, KK reductions on more complicated manifolds like spheres S^n (with $n > 1$) or so-called Calabi-Yau manifolds have been considered. For such reductions, the issue of consistency of the truncation of the massive KK modes is much more tricky. By now, it is reasonably well understood that sphere reductions are consistent. For Calabi-Yau manifolds, there is at present no hope of understanding this issue [9] and practitioners of Calabi-Yau reductions simply sweep the question of consistency under a big carpet, in the hope that the cleaning personnel doesn't dig it up.

3 The massless particle action in a KK background

Above, I mentioned that the KK vector $\mathcal{A}_\mu$ couples to massive KK modes, in the sense that these modes are charged under the abelian gauge transformation of $\mathcal{A}_\mu$. Any derivative of such a massive mode thus has to be covariantized with $\mathcal{A}_\mu$. Since we are mostly concerned with reductions in which all massive modes

[9] For reductions on S^1 we were able to derive the low-energy effective theory from the higher-dimensional one. Such an explicit derivation is at the moment not possible for reductions on Calabi-Yau manifolds. Rather, writing down the low-energy effective theory for such a reduction relies on educated guesswork, based on e.g., the number of supersymmetries that are preserved in such a reduction, what kind of massless modes one expects, and the structure of lower-dimensional supergravity theories.

are discarded, we did not go through the details of this reasoning, so the statement that $\mathcal{A}_\mu$ couples to the massive modes might not be too palatable. Here, we will present a different argument that should hopefully alleviate any concerns you might understandably have about the role of the KK vector in life, the universe and everything. Besides being quite cute, this demonstration also contains a couple of technical points that are often disregarded or even incorrectly treated in the literature and that are useful to point out. [10]

In short, what we will do is consider the worldline action that describes the dynamics of a massless particle moving in a $(D+1)$-dimensional spacetime with a compact S^1-direction. We will take the metric of this spacetime to be given by the KK Ansatz (2.8), so that we can try to see what the massless particle action looks like from the D-dimensional point of view. We will argue that we can consistently restrict this action to that of a particle that has fixed momentum in the z-direction, i.e., that of a particle that is massive when seen from D dimensions. Imposing this restriction and doing some manipulations, we will see that one indeed ends up with the action of a massive D-dimensional particle, coupled to the KK vector.

Let us start from the worldline action of a massless particle that moves in a $(D+1)$-dimensional spacetime with metric $\hat{g}_{MN}$:

$$S_{m=0}[X^M, e] = \frac{1}{2}\int \mathrm{d}\tau\, e^{-1}\hat{g}_{MN}(X^R)\dot{X}^M\dot{X}^N\,, \qquad\qquad \text{with}\ \ \dot{X}^M = \frac{\mathrm{d}}{\mathrm{d}\tau}X^M\,. \tag{3.1}$$

The justification for this action lies in the fact that it leads to the correct equation of motion (namely the geodesic equation) for the particle. More info on this action, and in particular a derivation of it as the massless limit of the action for a massive particle, can be found in appendix B. As before, we assume that our spacetime has a compact direction that takes the shape of a circle S^1 with coordinate z. We accordingly split the X^M as [11]

$$X^M = \{X^\mu, X^z \equiv Z\}\,. \tag{3.2}$$

The field that describes the particle's position along the circle will thus be denoted by Z. We are interested in finding out what this particle looks like when seen from D dimensions. For this reason, we will assume that the particle coordinates X^M only couple to the z-independent components of $\hat{g}_{MN}$ and not to its massive KK excitations that are assumed to be unobservable for a lower-dimensional creature. We thus split the M index in (3.1) into μ and z and replace $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$ and $\hat{g}_{zz}$ by the KK Ansatz (2.8) to get:

$$S_{\text{KK-particle}}[X^\mu, Z, e] = \frac{1}{2}\int \mathrm{d}\tau\, e^{-1}\left[\mathrm{e}^{2\alpha\varphi(X^\sigma)}g_{\mu\nu}(X^\rho)\dot{X}^\mu\dot{X}^\nu + \mathrm{e}^{2\beta\varphi(X^\rho)}\left(\dot{Z} + \mathcal{A}_\mu(X^\nu)\dot{X}^\mu\right)^2\right]. \tag{3.3}$$

I have explicitly written $g_{\mu\nu}(X^\rho)$, $\mathcal{A}_\mu(X^\nu)$ and $\varphi(X^\mu)$ to stress that they depend on the particle coordinates X^μ. I'll stop being so pedantic in what follows, but keep in mind that this dependence is there. As an exercise, you should

Exercise 7:

derive the equations of motion for the fields X^μ, Z and e from the action (3.3). You should get

$$\text{for } e\ :\quad \tilde{g}_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\mu\dot{X}^\mu\right)^2 = 0\,, \tag{3.4a}$$

$$\begin{aligned}\text{for } X^\mu\ :\quad & e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\dot{X}^\mu\right) + \tilde{\Gamma}^\mu_{\nu\rho}\dot{X}^\nu\dot{X}^\rho - \beta\mathrm{e}^{2\beta\varphi}\tilde{g}^{\mu\nu}\partial_\nu\varphi\left(\dot{Z} + \mathcal{A}_\rho\dot{X}^\rho\right)^2 \\ &\qquad - \mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\rho\dot{X}^\rho\right)\tilde{g}^{\mu\nu}\dot{X}^\sigma\mathcal{F}_{\nu\sigma} = 0\,,\end{aligned} \tag{3.4b}$$

$$\text{for } Z\ :\quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\mu\dot{X}^\mu\right)\right) = 0\,. \tag{3.4c}$$

[10] On top of all that, it is also often crucial in some stringy logic, for instance in identifying the strong coupling limit of type IIA string theory with a theory whose low-energy limit is 11-dimensional supergravity. It is this argument that identifies bound states of D0-branes with the massive KK modes of the 11-dimensional supergravity multiplet.

[11] It's perhaps slight overkill, but I will distinguish between the $x^M = \{x^\mu, z\}$ as a set of coordinates on the spacetime and the $X^M = \{X^\mu, Z\}$ that are specifically used to denote just the coordinates of the particle. The latter correspond to dynamical fields in the worldline actions, so it makes sense to assign special symbols to them.

Here, $\tilde{g}_{\mu\nu} \equiv \mathrm{e}^{2\alpha\varphi} g_{\mu\nu}$, $\tilde{g}^{\mu\nu}$ is the inverse of $\tilde{g}_{\mu\nu}$ and $\tilde{\Gamma}^{\mu}_{\nu\rho}$ is the Levi-Civita connection constructed out of $\tilde{g}_{\mu\nu}$.[12] As before, we also have that $\mathcal{F}_{\mu\nu} \equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]}$.

Note that the field variable Z is cyclic, meaning that only $\dot{Z}$ appears in (3.3) and not Z itself. From classical mechanics, we then know that the momentum P_Z conjugate to Z:

$$P_Z \equiv \frac{\partial \mathcal{L}_{\text{KK-particle}}}{\partial \dot{Z}} = e^{-1}\mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right) , \tag{3.5}$$

is conserved. This conservation of P_Z is just expressed by the equation of motion (3.4c) of Z. The fact that P_Z is conserved (and that we are only considering a single particle) implies that we can consistently restrict ourselves to the dynamics of our particle with a fixed value of P_Z. Let us then focus on that sector of our particle theory with a fixed non-zero value of P_Z that we will call p:

$$e^{-1}\mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right) = p = \text{constant} \qquad \Leftrightarrow \qquad \left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right) = e\mathrm{e}^{-2\beta\varphi} p\,. \tag{3.6}$$

Since in this sector, the particle has non-zero momentum in the compact direction, it should correspond to a massive particle from the D-dimensional point of view. We can try to verify this. Let's first restrict the equations of motion (3.4) to the sector with $P_Z = p$, by replacing $\left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right)$ by $e\mathrm{e}^{-2\beta\varphi}p$. This gives:

$$\tilde{g}_{\mu\nu}\dot{X}^\mu \dot{X}^\nu + e^2 \mathrm{e}^{-2\beta\varphi} p^2 = 0\,, \tag{3.7a}$$

$$e\frac{\mathrm{d}}{\mathrm{d}\tau}\left(e^{-1}\dot{X}^\mu\right) + \tilde{\Gamma}^\mu_{\nu\rho}\dot{X}^\nu \dot{X}^\rho - \beta e^2 p^2 \mathrm{e}^{-2\beta\varphi}\tilde{g}^{\mu\nu}\partial_\nu\varphi - ep\tilde{g}^{\mu\nu}\dot{X}^\sigma \mathcal{F}_{\nu\sigma} = 0\,. \tag{3.7b}$$

This substitution in the equations of motion is of course allowed but the result is not so transparent. It would be nice if we could somehow do it in the action (3.3) and recognize the result as the action of a massive particle. Here, we however run into the problem that in general we are not allowed to substitute solutions of equations of motion into an action without running into an inconsistency! You can explore what goes wrong in the following exercise.

Exercise 8:

Substituting $\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu$ by $e\mathrm{e}^{-2\beta\varphi}p$ in (3.3) gives

$$S'[e, X^\mu] = \frac{1}{2}\int \mathrm{d}\tau\, e^{-1}\left(\mathrm{e}^{2\alpha\varphi} g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu + e^2 \mathrm{e}^{-2\beta\varphi}p^2\right) . \tag{3.8}$$

Derive the equations of motion of e and X^μ from S' and show that they are not the same as (3.7).
We can now say a word about the word "inconsistency". One can do two things. First, one can do the substitution in the equations of motion of (3.3), to obtain the equations (3.7) for X^μ and e that no longer contain Z (one no longer has to consider the equation of motion of Z, since it is identically fulfilled by the substitution). Secondly, one can first do the substitution in (3.3), giving the new action S' that only depends on X^μ and e, and then derive the equations of motion of X^μ and e from this new action S'. The word "inconsistency" here means that these two procedures do not give the same result!

Luckily, there exists a way out and one can rewrite the action (3.3) in a way that allows to do the substitution $\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu \to e\mathrm{e}^{-2\beta\varphi}p$ in a consistent manner.[13] The trick is to rewrite (3.3) in "Hamiltonian" form, where the quotes indicate that we will only perform the passage to a first-order Hamiltonian form for the Z variable and not for the X^μ variables. As a first step, we pass from $(Z, \dot{Z})$ variables to (Z, P_Z) variables via the formulas

$$P_Z = e^{-1}\mathrm{e}^{2\beta\varphi}\left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right) \qquad \Leftrightarrow \qquad \dot{Z} = e\mathrm{e}^{-2\beta\varphi}P_Z - \mathcal{A}_\mu \dot{X}^\mu\,, \tag{3.9}$$

[12] I know, I know, footnote 2,... I only said "I will *try* to stick to just hats" there.

[13] What we will be doing here is a simple example of what is called "Hamiltonian reduction".

and we define the "Hamiltonian" by Legendre transforming in the Z-variable as usual:

$$\mathcal{H}[X^\mu, \dot{X}^\mu, Z, P_Z, e] \equiv P_Z \dot{Z} - \mathcal{L}[X^\mu, \dot{X}^\mu, Z, \dot{Z}, e]\,, \tag{3.10}$$

where on the right-hand-side it is understood that $\dot{Z}$ stands for its expression in terms of P_Z: $\dot{Z} = e\mathrm{e}^{-2\beta\varphi} P_Z - \mathcal{A}_\mu \dot{X}^\mu$. The Lagrangian $\mathcal{L}[X^\mu, \dot{X}^\mu, Z, \dot{Z}, e]$ is that of the action (3.3):

$$\mathcal{L}[X^\mu, \dot{X}^\mu, Z, \dot{Z}, e] = \frac{1}{2} e^{-1} \left[\mathrm{e}^{2\alpha\varphi} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu + \mathrm{e}^{2\beta\varphi} \left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right)^2\right]. \tag{3.11}$$

Exercise 9:

Show that $\mathcal{H}$ is explicitly given by

$$\mathcal{H}[X^\mu, \dot{X}^\mu, Z, P_Z, e] = \frac{1}{2} e\mathrm{e}^{-2\beta\varphi} P_Z^2 - \frac{1}{2} e^{-1} \mathrm{e}^{2\alpha\varphi} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - \mathcal{A}_\mu \dot{X}^\mu P_Z\,. \tag{3.12}$$

We can now rewrite (3.3) in "Hamiltonian" form as:

$$\begin{aligned} \tilde{S}[X^\mu, Z, P_Z, e] &= \int \mathrm{d}\tau\, \left(P_Z \dot{Z} - \mathcal{H}[X^\mu, \dot{X}^\mu, Z, P_Z, e]\right) \\ &= \frac{1}{2} \int \mathrm{d}\tau\, e^{-1} \left(\mathrm{e}^{2\alpha\varphi} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - e^2 \mathrm{e}^{-2\beta\varphi} P_Z^2\right) + \int \mathrm{d}\tau\, P_Z \left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right). \end{aligned} \tag{3.13}$$

The equation of motion of Z immediately tells us that P_Z is conserved: $\dot{P}_Z = 0$. The equation of motion of P_Z reproduces (3.5). Substituting P_Z by this expression (3.5) in terms of $\dot{Z}$ then immediately gives back (3.3).[14] The actions (3.3) and (3.13) are thus equivalent.

Working with (3.13) has the nice feature that we can consistently focus on the sector with fixed non-zero P_Z. Setting $P_Z = p$ in (3.13) gives

$$\begin{aligned} \tilde{S}[X^\mu, e] &= \frac{1}{2} \int \mathrm{d}\tau\, e^{-1} \left(\mathrm{e}^{2\alpha\varphi} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - e^2 \mathrm{e}^{-2\beta\varphi} p^2\right) + p \int \mathrm{d}\tau\, \left(\dot{Z} + \mathcal{A}_\mu \dot{X}^\mu\right) \\ &= \frac{1}{2} \int \mathrm{d}\tau\, e^{-1} \left(\tilde{g}_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - e^2 \mathrm{e}^{-2\beta\varphi} p^2\right) + p \int \mathrm{d}\tau\, \mathcal{A}_\mu \dot{X}^\mu\,, \end{aligned} \tag{3.14}$$

where in the second step we dropped the $\dot{Z}$ term, since it is just a total derivative. This procedure is consistent, as you are asked to show in the following exercise.

Exercise 10:

Derive the equations of motion of (3.14) and show that they coincide with (3.7).

The action (3.14) almost looks like something we know. To make it a bit more recognizable, let us note that we can solve e from its own equation of motion as follows:

$$e = |p|^{-1} \mathrm{e}^{(\alpha+\beta)\varphi} \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}\,. \tag{3.15}$$

We are allowed to plug this solution in (3.14) without being punished by the consistency police (see footnote 14). Doing this then finally gives

$$S[X^\mu] = -|p| \int \mathrm{d}\tau\, \sqrt{-g'_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} + p \int \mathrm{d}\tau\, \mathcal{A}_\mu \dot{X}^\mu\,, \qquad \text{with } g'_{\mu\nu} = \mathrm{e}^{2(\alpha-\beta)\varphi} g_{\mu\nu}\,. \tag{3.16}$$

[14] As seen in exercise 8, substituting solutions of equations of motion in actions is in general not consistent in the sense explored in that exercise! It is however consistent if the solution for the field that is being substituted is obtained from that field's own equation of motion, as is the case for P_Z here!

Tadaa! This is the action of a massive particle that moves in a D-dimensional background with metric $g'_{\mu\nu}$ [15] and that is moreover charged under the KK vector $\mathcal{A}_\mu$, just as we set out to show! Note that the $|p|$ in front of the first term is the mass M of the particle, while the p in front of the second term is the charge of the particle. Our particle thus obeys the remarkable equality:

$$\text{mass} = |\text{charge}| \,. \tag{3.17}$$

This equality is known as the Bogomol'nyi-Prasad-Sommerfield or BPS equality, a term that you are likely going to encounter a lot. Because it obeys this BPS equality, our particle is also called a BPS particle or is said to "saturate the BPS bound". [16]

A Reduction of general coordinate transformations

All components of $\hat{\xi}^M$ are independent of each other, so we can restrict ourselves to the situation in which $\hat{\xi}^\mu = 0$ and only $\hat{\xi}^z$ is non-zero. Doing this in (2.5) and using that $\partial_z \hat{g}_{MN} = 0$, one ends up with

$$2\partial_z \partial_{(M} \hat{\xi}^z \hat{g}_{N)z} = 0 \,. \tag{A.1}$$

Take first $M = N = z$:

$$2\partial_z^2 \hat{\xi}^z \hat{g}_{zz} = 0 \qquad \Rightarrow \qquad \partial_z^2 \hat{\xi}^z = 0 \,. \tag{A.2}$$

From this, one sees that

$$\hat{\xi}^z = c(x^\mu) z + \lambda(x^\mu) \,. \tag{A.3}$$

Next, take $M = \mu$ and $N = z$ in (A.1). Using that $\partial_z^2 \hat{\xi}^z = 0$, one gets

$$\partial_z \partial_\mu \hat{\xi}^z \hat{g}_{zz} = 0 \qquad \Rightarrow \qquad \partial_z \partial_\mu \hat{\xi}^z = 0 \,. \tag{A.4}$$

Plugging the form (A.3) in this requirement, one finds that

$$\partial_\mu c(x^\nu) = 0 \qquad \Rightarrow \qquad c(x^\mu) = c = \text{constant} \,. \tag{A.5}$$

All in all, one thus finds that $\hat{\xi}^z$ has to be of the form

$$\hat{\xi}^z = cz + \lambda(x^\mu) \,, \tag{A.6}$$

with c a constant.

B The massless particle action

The kinematics of a massive particle that moves in a space-time with metric $\hat{g}_{MN}$ is specified by giving the particle's coordinates X^M as functions $X^M(\tau)$ of a parameter τ. As the so-called worldline parameter τ varies, the functions $X^M(\tau)$ trace out the path or worldline of the particle in spacetime. The dynamics of a massive particle in a space-time with metric $\hat{g}_{MN}$ is given by the "Nambu-Goto"-like worldline action:

$$S_{\text{NG}}[X^M] = -m \int \mathrm{d}\tau \sqrt{-\hat{g}_{MN}(X^R)\dot{X}^M \dot{X}^N} \,, \qquad \dot{X}^M = \frac{\mathrm{d}}{\mathrm{d}\tau} X^M \,. \tag{B.1}$$

[15] At this point, I regret some of my choices in life, in particular having written footnote 2...

[16] The term "BPS bound" comes from the study of QFTs with extended supersymmetry. There, one can show that all states of a supermultiplet, i.e., a set of particle states that are related by supersymmetry, satisfy the inequality mass $\geq$ |charge|. The mass of the states of a supermultiplet is thus bounded from below by their charge. It turns out that when the inequality is saturated, so when mass = |charge|, there are less states in a supermultiplet than when the inequality is a strict one. This simple fact actually allows one to extract a lot of information about supersymmetric QFTs without having to rely on perturbation theory!

Here, m is the mass of the particle and we have written $S_{\rm NG}[X^M]$ to emphasize that the dynamical fields of this action are the $X^M(\tau)$. In case the particle also has an electric charge q, this action contains an extra term

$$S[X^M] = -m\int \mathrm{d}\tau\,\sqrt{-\hat{g}_{MN}(X^R)\dot{X}^M\dot{X}^N} - q\int \mathrm{d}\tau \hat{A}_M(X^N)\dot{X}^M\,, \tag{B.2}$$

where $\hat{A}_M$ is the vector potential that gives rise to the electric and magnetic fields in which the particle moves. We will take $q=0$ for now.

The action (B.1) vanishes as $m \to 0$ and is thus pretty useless to describe the dynamics of a massless particle. This can however be remedied by introducing an extra, auxiliary field $e(\tau)$ and considering the following "Polyakov"-like action (for the dynamical variables X^M and e):

$$S_{\rm P}[e, X^M] = \frac{1}{2}\int \mathrm{d}\tau\,\left(e^{-1}\hat{g}_{MN}(X^R)\dot{X}^M\dot{X}^N - em^2\right)\,. \tag{B.3}$$

Although the actions $S_{\rm NG}$ and $S_{\rm P}$ look pretty different, they are equivalent. To see this, we note that the equation of motion of e that is derived from (B.3) is given by:

$$-e^{-2}\hat{g}_{MN}\dot{X}^M\dot{X}^N - m^2 = 0\,. \tag{B.4}$$

This equation is algebraic in e and can thus be used to solve for e. One finds:

$$e = \frac{1}{m}\sqrt{-\hat{g}_{MN}\dot{X}^M\dot{X}^N}\,. \tag{B.5}$$

Since we obtained this solution for e from its own equation of motion, we can consistently substitute it in (B.3) and doing this leads to the action (B.1).

Exercise 11:

Go through these steps.

Even though $S_{\rm P}$ and $S_{\rm NG}$ are equivalent as far as the dynamics of a massive particle is concerned, $S_{\rm P}$ has one major advantage: it has a non-trivial $m \to 0$ limit. Taking this limit, we thus obtain the following worldline action for a massless particle:

$$S_{m=0}[X^M, e] = \frac{1}{2}\int \mathrm{d}\tau\, e^{-1}\hat{g}_{MN}(X^R)\dot{X}^M\dot{X}^N\,. \tag{B.6}$$